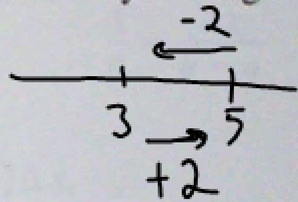
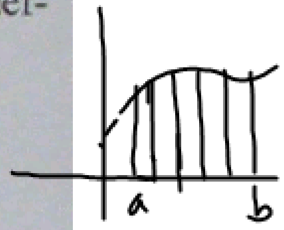
Properties of Definite Integrals

In defining $\int_a^b f(x)$ as a limit of sums $\sum c_k \Delta x_k$, we moved from left to right across the interval $[a, b]$. What would happen if we integrated in the *opposite direction*? The integral would become $\int_b^a f(x)\,dx$—again a limit of sums of the form $\sum f(c_k)\Delta x_k$—but this time each of the Δx_k's would be negative as the x-values *decreased* from b to a. This would change the signs of all the terms in each Riemann sum, and ultimately the sign of the definite integral. This suggests the rule

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

$\int_a^a f(x)\,dx = 0$



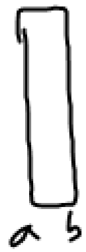


b-a

Since the original definition did not apply to integrating backwards over an interval, we can treat this rule as a logical extension of the definition.

Although $[a, a]$ is technically not an interval, another logical extension of the definition is that $\int_a^a f(x)\,dx = 0$.

These are the first two rules in Table 5.3. The others are inherited from rules that hold for Riemann sums. However, the limit step required to *prove* that these rules hold in the limit (as the norms of the partitions tend to zero) places their mathematical verification beyond the scope of this course. They should make good sense nonetheless.



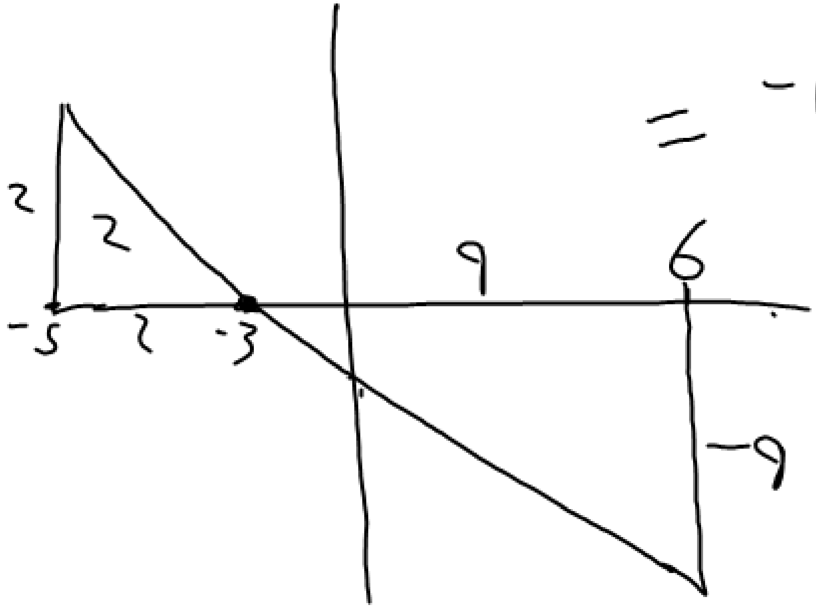
b-a = pos

a-b = neg

$$\int_{-5}^{6} \frac{9-x^2}{x-3}\,dx = \frac{(3-x)(3+x)}{(x-3)}$$

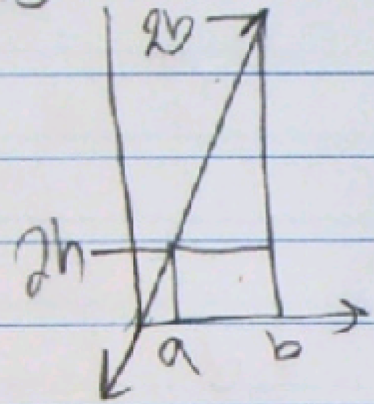
$$= \frac{-\cancel{(x-3)}(3+x)}{\cancel{(x-3)}}$$

$$= -(3+x) = -x-3$$



$$.5(2^2 - 9^2) = -38.5$$

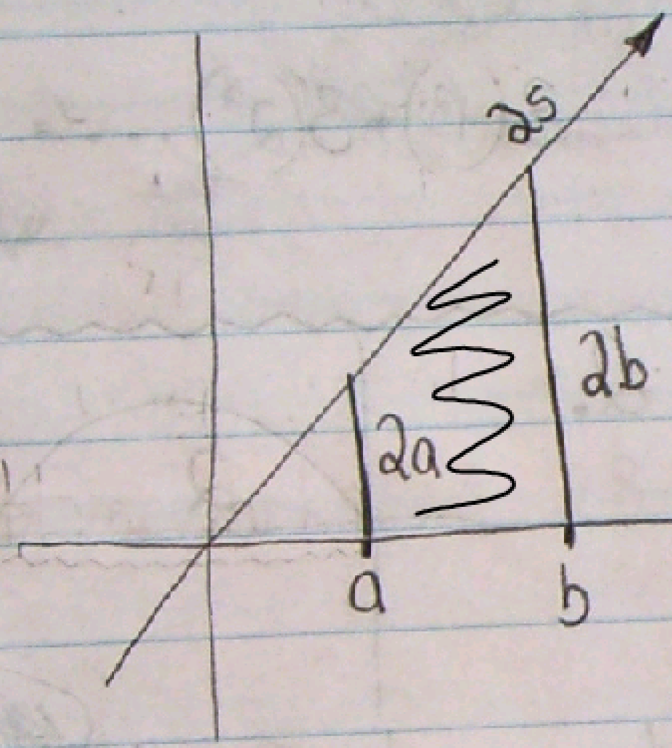
25. $\int_a^b 2s\,ds \qquad 0<a<b$



$\frac{1}{2}(b(2b)) - \frac{1}{2}(a)(2a) = A$

$b^2 - a^2 = A$

Area $= b^2 - a^2$



$$\text{Area} = \tfrac{1}{2}(b_1 + b_2) \cdot \text{height}$$

$$A = \tfrac{1}{2}(2a + 2b) \cdot (b - a)$$

$$(a + b)(b - a)$$

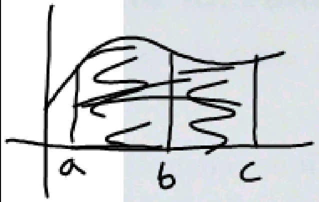
$$b^2 - a^2$$

1. *Order of Integration:* $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ A definition

2. *Zero:* $\displaystyle\int_a^a f(x)\,dx = 0$ Also a definition

3. *Constant Multiple:* $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ Any number k

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx \qquad k = -1$$



4. *Sum and Difference:* $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

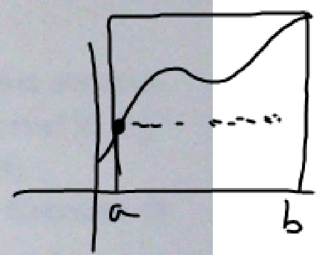
$$\frac{d}{dx}\left(x^2+5x\right)$$

$$\frac{dx^2}{dx} + \frac{d\,5x}{dx}$$

5. *Additivity:* $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

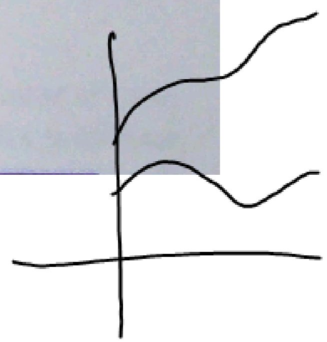
6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$



7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$$f(x) \ge 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0 \qquad g = 0$$



Suppose that f and g are continuous functions and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.3 to find each integral.

(a) $\int_2^2 g(x)\,dx$ 0 [redacted]

(b) $\int_5^1 g(x)\,dx$ -8 [redacted]

(c) $\int_1^2 3f(x)\,dx$ -12 [redacted]

(d) $\int_2^5 f(x)\,dx$ 10 [redacted]

(e) $\int_1^5 [f(x) - g(x)]\,dx$ -2 [redacted]

(f) $\int_1^5 [4f(x) - g(x)]\,dx$ 16 [redacted]

EXAMPLE 2 Finding Bounds for an Integral

Show that the value of $\int_0^1 \sqrt{1 + \cos x}\, dx$ is less than 3/2.

SOLUTION

The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x)\, dx$ and that $\max f \cdot (b - a)$ is an *upper bound.* The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{2}$, so

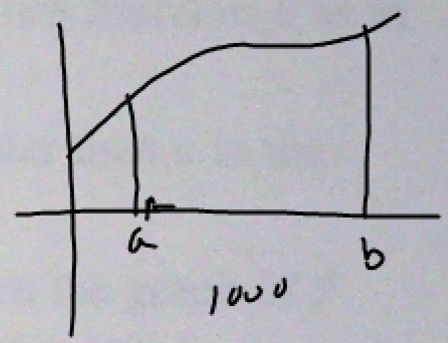
$$\int_0^1 \sqrt{1 + \cos x}\, dx \le \sqrt{2} \cdot (1 - 0) = \sqrt{2}.$$

Since $\int_0^1 \sqrt{1 + \cos x}\, dx$ is bounded above by $\sqrt{2}$ (which is 1.414...), it is less than 3/2.

Now try Exercise 7.

Consider, then, what happens if we take a large *sample* of n numbers from regular subintervals of the interval $[a, b]$. One way would be to take some number c_k from each of the n subintervals of length

$$\frac{\Delta x}{b-a} = \frac{b-a}{n}. \div (b-a)$$



The average of the n sampled values is

$$\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} = \frac{1}{n} \cdot \sum_{k=1}^{n} f(c_k)$$

$$\frac{b-a}{1000}$$

$$= \frac{\Delta x}{b-a} \sum_{k=1}^{n} f(c_k) \qquad \frac{1}{n} = \frac{\Delta x}{b-a}$$

$$= \frac{1}{b-a} \cdot \sum_{k=1}^{n} f(c_k)\Delta x.$$

Does this last sum look familiar? It is $1/(b-a)$ times a Riemann sum for f on $[a, b]$. That means that when we consider this averaging process as $n \to \infty$, we find it *has a limit*, namely $1/(b-a)$ times the integral of f over $[a, b]$. We are led by this remarkable fact to the following definition.

DEFINITION Average (Mean) Value

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Do Exploration 1 - p. 289

$$\frac{1}{b-a}\cdot\underbrace{\int_a^b f(x)\,dx}_{\text{area}}$$

$$\frac{1}{r--r}\cdot\pi r^2$$

$$\frac{1}{2r}\cdot\pi r^2 = \frac{\pi r}{2}$$

$\boxed{\text{HW}}$

Read rest 5.3

#2-6 (2), 7, 8, 11-14 (2), 15 or 16, 17 or 18